Scan Conversion 2

Scan Converting Circles

Version 1: really bad
For $x = -R$ to $R$

$y = \sqrt{R^2 - x^2}$;
Pixel ($\text{round}(x)$, $\text{round}(y)$);
Pixel ($\text{round}(x)$, $\text{round}(-y)$);

Version 2: slightly less bad
For $x = 0$ to $360$

Pixel ($\text{round}(R \cdot \cos(x))$, $\text{round}(R \cdot \sin(x))$);

Version 3 — Use Symmetry

- Symmetry: If $(x_0 + a, y_0 + b)$ is on circle
  - also $(x_0 \pm a, y_0 \pm b)$ and $(x_0 \pm b, y_0 \pm a)$;
  hence 8-way symmetry.
- Reduce the problem to finding the pixels for 1/8 of the circle

Using the Symmetry

- Scan top right 1/8 of circle of radius $R$
- Circle starts at $(x_0, y_0 + R)$
- Let’s use another incremental algorithm with decision variable evaluated at midpoint
**Sketch of Incremental Algorithm**

\[ x = x_0; y = y_0 + R; \text{Pixel}(x, y); \]

```
for (x = x_0 + 1; (x - x_0) > (y - y_0); x++) {
    if (decision_var < 0) {
        /* move east */
        update decision_var;
    } else {
        /* move south east */
        update decision_var;
        y--;
    }
    Pixel(x, y);
}
```

- Note: can replace all occurrences of \( x_0, y_0 \) with 0, 0 and \( \text{Pixel}(x_0 + x, y_0 + y) \) with \( \text{Pixel}(x, y) \)
- Shift coordinates by \((-x_0, -y_0)\)

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**What we need for Incremental Algorithm**

- Decision variable
  - negative if we move E, positive if we move SE (or vice versa).
- Follow line strategy: Use implicit equation of circle
  \[ f(x, y) = x^2 + y^2 - R^2 = 0 \]
  \( f(x, y) \) is zero on circle, negative inside, positive outside
- If we are at pixel \((x, y)\)
  - examine \((x + 1, y)\) and \((x + 1, y - 1)\)
- Compute \( f \) at the midpoint

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**Decision Variable**

- Evaluate \( f(x, y) = x^2 + y^2 - R^2 \) at the point \( \left(x + \frac{1}{2}, y - \frac{1}{2}\right) \)
- We are asking: “Is \( f \left(x + \frac{1}{2}, y - \frac{1}{2}\right) = (x + 1)^2 + (y - 1)^2 - R^2 \) positive or negative?” (It is zero on circle)
- If **negative**, midpoint inside circle, choose E
  - vertical distance to the circle is less at \((x + 1, y)\) than at \((x + 1, y - 1)\).
- If **positive**, opposite is true, choose SE

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**The right decision variable?**

- Decision based on vertical distance
- Ok for lines, since \( d \) and \( d_{vert} \) are proportional
- For circles, not true:
  \[ d((x + 1, y, \text{Circ}) = \sqrt{(x+1)^2 + y^2} - R \]
  \[ d((x + 1, y - 1, \text{Circ}) = \sqrt{(x+1)^2 + (y-1)^2} - R \]
  - Which \( d \) is closer to zero? (i.e. which of the two values below is closer to \( R \):
    \[ \sqrt{(x+1)^2 + y^2} \] or \[ \sqrt{(x+1)^2 + (y-1)^2} \]
Alternate Phrasing (1/3)

- We could ask instead: "Is \((x + 1)^2 + y^2\) or \((x + 1)^2 + (y - 1)^2\) closer to \(R^2\)?"

- The two values in equation above differ by 
\[
[(x + 1)^2 + y^2] - [(x + 1)^2 + (y - 1)^2] = 2y - 1
\]

Alternate Phrasing (2/3)

- The second value, which is always less, is closer if its difference from \(R^2\) is less than 
\[
\left( \frac{1}{2} \right) (2y - 1)
\]
i.e., if
\[
R^2 - [(x + 1)^2 + (y - 1)^2] < \frac{1}{4} (2y - 1)
\]
then
\[
0 < y - \frac{1}{2} (x + 1)^2 + (y - 1)^2 - R^2
\]
so
\[
0 < (x + 1)^2 + y^2 - 2y + 1 + y - \frac{1}{2} - R^2
\]
so
\[
0 < (x + 1)^2 + y^2 - y + \frac{1}{2} - R^2
\]
so
\[
0 < (x + 1)^2 + \left( y - \frac{1}{2} \right)^2 + \frac{1}{4} - R^2
\]

Alternate Phrasing (3/3)

- The radial distance decision is whether
\[
d_1 = (x + 1)^2 + \left( y - \frac{1}{2} \right)^2 + \frac{1}{4} - R^2
\]
is positive or negative

- And the vertical distance decision is whether
\[
d_2 = (x + 1)^2 + \left( y - \frac{1}{2} \right)^2 - R^2
\]
is positive or negative; \(d_1\) and \(d_2\) are \(\frac{1}{4}\) apart.

- The integer \(d_1\) is positive only if \(d_2 + \frac{1}{4}\) is positive (except special case where \(d_2 = 0\)).

Incremental Computation, Again (1/2)

- How to compute the value of
\[
f(x, y) = (x + 1)^2 + \left( y - \frac{1}{2} \right)^2 - R^2
\]
at successive points?

- Answer: Note that
\[
f(x + 1, y) - f(x, y)
\]
is just
\[
\Delta_E(x, y) = 2x + 3
\]
and that
\[
f(x + 1, y - 1) - f(x, y)
\]
is just
\[
\Delta_{SE}(x, y) = 2x + 3 - 2y + 2
\]
Incremental Computation (2/2)

- If we move E, update by adding $2x + 3$
- If we move SE, update by adding $2(x-y) + 5$
- Forward differences of a 1st degree polynomial are constants and those of a 2nd degree polynomial are 1st degree polynomials – this “first order forward difference,” like a partial derivative, is one degree lower

Second Differences (1/2)

- The function $\Delta_E(x,y) = 2x + 3$ is linear, hence amenable to incremental computation:
  \[
  \Delta_E(x+1,y) - \Delta_E(x,y) = 2 \\
  \Delta_E(x+1,y-1) - \Delta_E(x,y) = 2 \\
  \]

- Similarly
  \[
  \Delta_{SE}(x+1,y) - \Delta_{SE}(x,y) = 2 \\
  \Delta_{SE}(x+1,y-1) - \Delta_{SE}(x,y) = 4 \\
  \]

Midpoint Eighth Circle Algorithm

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Analysis

- Uses floats!
- 1 test, 3 or 4 additions per pixel
- Initialization can be improved
- Multiply everything by 4 → No Floats!
  - Makes the components even, but sign of decision variable remains same

Questions

- Are we getting all pixels whose distance from the circle is less than ½?
- Why is \( y > x \) the right stopping criterion?
- What if it were an ellipse?

Patterned Lines

- Patterned line from \( P \) to \( Q \) is not same as patterned line from \( Q \) to \( P \).
- Patterns can be geometric or cosmetic
  - Cosmetic: Texture applied after transformations
  - Geometric: Pattern subject to transformations

Geometric Pattern vs. Cosmetic Pattern

- Geometric (Perspectivized/Filtered)
- Cosmetic (Contact Paper)
Aligned Ellipses

- Equation is
  \[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

  i.e.,
  \[ b^2x^2 + a^2y^2 = a^2b^2 \]

- Computation of \( \Delta_e \) and \( \Delta_w \) is similar
- Only 4-fold symmetry
- When do we stop stepping horizontally and switch to vertical?

Direction Changing Criterion (1/2)

- When absolute value of slope of ellipse is more than 1:
  \[ \frac{\partial f}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y) > 0 \]

  \[ \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \end{bmatrix} \]
  - This vector points more right than up when
  - How do you check this? At a point \((x,y)\) for which \(f(x,y) = 0\), a vector perpendicular to the level set is \(\nabla f(x,y)\) which is

Direction Changing Criterion (2/2)

- In our case,
  \[ \frac{\partial f}{\partial x}(x,y) = 2a^2x \]
  and
  \[ \frac{\partial f}{\partial y}(x,y) = 2b^2y \]

  so we check for
  \[ 2a^2x - 2b^2y > 0 \]
  i.e.
  \[ a^2x - b^2y > 0 \]

  - This, too, can be computed incrementally

Problems with Aligned Ellipses

- Now in ENE octant, not ESE octant

- This problem is artifact of aliasing – much more on this later
Non-Integer Primitives and General Conics

- **Non-Integer Primitives**
  - Initialization is harder
  - Endpoints are hard, too
    - making Line \((P, Q)\) and Line \((Q, R)\) join properly is a good test
  - Symmetry is lost

- **General Conics**
  - Very hard--the octant-changing test is tougher, the difference computations are tougher, etc.
    - do it only if you have to.
  - Note that drawing gray-scale conics is easier than drawing B/W conics

Generic Polygons

(More information and these pictures on page 92-93 of textbook)