The Biography of Tien-Yien Li (1945 -)

Jiu Ding

Mr. Tien-Yien Li, of Hunan ancestry, was born in June of 1945 in Sha County, Fujian Province of China. His father, Ding-Xun Li, studied overseas at the medical school of Tokyo Imperial University and received a medical doctorate degree. In 1934, he returned to China and became a faculty member of Hunan Xiang-ya Medical School. In 1939, he served as president of Fujian Provincial Medical College. At age three, Tien-Yien Li followed his parents to Taiwan, where he received his education until he graduated from college. He graduated from the Department of Mathematics at the National Tsinghua University in 1968, the first mathematics graduating class of the University. After a year of mandatory military service, he went to the U.S. and attended the University of Maryland to study for his PhD in mathematics. He received his doctorate in 1974 under the guidance of James A. Yorke.

Tien-Yien Li was an instructor in the mathematics department of the University of Utah from 1974 to 1976. From 1976 to present, he taught at Michigan State University, three years as an assistant professor and four years as an associate professor. He has been a full professor since 1983. In 1998, he received the title of University Distinguished Professor. Tien-Yien Li was a Visiting Associate Professor at the Mathematical Research Center of the University of Wisconsin at Madison from 1978 to 1979. He has been a Guest Professor at Jilin University (since 1987) and Tsinghua University in Beijing (since 1991). As a Visiting Professor, he worked at Kyoto University's Research Institute of Mathematical Sciences(1987-1988), the Mathematical Sciences Research Institute at the University of California, Berkeley (1998), and the City University of Hong Kong (2000). In the summer of 1997, he was a Senior Research Member in the Advanced Theoretical Science Research

Center at the Tsinghua University in Beijing.

Tien-Yien Li, despite his numerous ailments, has been a trailblazer in several important fields of Applied Mathematics and Computational Mathematics. Some of his monumental accomplishments include: he and Yorke's paper, "Period three implies chaos," first introduced the concept of "chaos" in the field of mathematics; his proof of Ulam's conjecture is the fundamental work in the computation of invariant measures of dynamical systems; his idea and numerical method with R. B. Kellogg and J. A. Yorke in computing Brouwer's fixed point opened a new era for the research in modern homotopy continuation methods. His extensive and deep research with his collaborators as well as his students on the algebraic eigenvalue problem and multivariate polynomial systems has earned him the honor of being one of the world's leaders in the field.

Tien-Yien Li received the highly prestigious Guggenheim Fellowship in 1995, Michigan State University's Distinguished Faculty Award as well as Frame Teaching Award in 1996, and College of Sciences Distinguished Alumni Award of National Tsinghua University in Taiwan in 2002.

1 "Period Three Implies Chaos"

Today, anyone who understands a little about dynamical systems would know Li and Yorke's extremely important paper "Period three implies chaos" appeared in the American Mathematical Monthly in 1975. This article practically invented the term of chaos and turned over a new leaf in the research of chaotic dynamical systems.

In the world of science, the discovery of chaos, the theory of relativity, and quantum mechanics are considered the three monumental discoveries of the twentieth century. As early as the end of the nineteenth century and the beginning of the twentieth century, the great French mathematician H. Poincaré already knew the sensitive dependence on initial conditions of solutions of Newton's differential equations for motions upon studying the "three-body problem" in celestial mechanics. In the early 1960s, Professor Edward N. Lorenz of the Massachusetts Institute of Technology's meteorology department used three simple ordinary differential equations to describe a convection-diffusion problem that can be applied to weather prediction, and he accidentally discovered the impossibility of long-term weather forecast, which is referred to as the "butterfly effect." A decade later, Robert M. May, a biology professor of Princeton University, utilized the logistic model $S(x) = \alpha x(1 - x)$ for studying the species populations, and he surprisingly found that when the number α approaches 4, the iteration sequence $\{S^n(x)\}$ becomes increasingly complicated. With these scientific researches as background, a mathematical definition of chaos emerged in Li and Yorke's famous paper.

In 1972, Lorenz's four papers concerning a weather prediction model attracted the attention of Professor Yorke in the Institute of Fluid Dynamics and Applied Mathematics (it's now called the Institute of Physical Sciences and Technology) at the University of Maryland and his PhD student Tien-Yien Li. When Tien-Yien Li appeared in Yorke's office one afternoon in March, 1973, Yorke said to him, "I have a good idea for you." This idea has evolved in Yorke's head, yet he has not been able to prove it completely. Two weeks later, Tien-Yien Li, skillfully manipulating his calculus techniques, proved what is later known as the famous Li-Yorke Theorem: if a continuous real function f on the real axis R has a point of period 3, that is there is a point a such that $f(a) = b$, $f(b) = c$, and $f(c) = a$, where $a \neq b \neq c$, then (i) f has a point of period n for any positive integer n; (ii) there is an uncountable subset S of R such that for any two points $x \neq y$ in S, the number sequence $|f^n(x) - f^n(y)|$ has a convergent subsequence that converges to 0 and a convergent subsequence that converges to a positive number. Moreover, for any periodic point p of f and any $x \in S$, the sequence $|f^n(p) - f^n(x)|$ has a convergent subsequence that converges to a positive number. After they finished it, according to Yorke's intention, the paper was sent to the highly acclaimed American Mathematical Monthly. However, the paper was rejected because it does not appeal to the current issues' pool of college readers. The editor agreed that the paper may be re-submitted if the authors could rewrite the paper so that average college students can understand it. Since Tien-Yien Li's busied himself with research on differential equations and the others, this paper sat untouched on his desk for nearly a year.

The year 1974 is a "special year" of biomathematics in the department of mathematics at the University of Maryland. In this year, top scholars in the field were invited to give lectures every week. In the first week of May, the department invited Professor Robert May of Princeton University to lecture for a week. On the last day, he lectured about the logistic model $S(x) = \alpha x(1 - x)$ and reported on its iteration sequences' complicated dynamical behavior as the parameter α is near 4, yet he did not offer an explanation, thinking that the phenomenon is perhaps caused by computation errors. After Yorke heard this lecture, he gave Professor May the paper of the Li-Yorke Theorem, which had rested on the desk for nearly a year, on their way to the airport. May was stunned upon reading the conclusion to the paper, and he recognized that this theorem had fully explained his uncertainties. At once, Yorke returned from the airport and contacted Li, "We should rewrite this paper immediately." The task was completed within two weeks, and it was accepted by the American Mathematical Monthly. It appeared in the December issue of 1975.

The paper entitled "Period three implies chaos" first strictly introduced a mathematical definition of chaos. Even though the former Soviet mathematician A. N. Sharkovsky proved the equivalent of the first part of the Li-Yorke Theorem, yet only the latter part of the Li-Yorke Theorem thoroughly unveiled the nature and characteristics of chaos: the sensitive dependence on initial conditions and the resulting unpredictable nature of the eventual behavior of the dynamics. Statistics show that this particular paper is one of the most frequently cited papers in mathematics and physics. Up until 2001, it had been cited more than 700 times.

2 Ulam's Conjecture

Ergodic theory is a mathematical branch of statistical research that concerns nonlinear dynamical systems; it is a composite field that combines measure theory, functional analysis, topology, modern algebra, etc. It is widely used in physical and engineering sciences such as statistical physics and electrical circuits. An important topic in ergodic theory concerns the existence and computation of an absolutely continuous invariant measure associated with a nonlinear mapping. This problem is reduced to the existence and computation of a fixed density function of the corresponding Frobenius-Perron operator defined on a space of Lebesgue integrable functions (i.e., an L^1-space). To the chaotic dynamical systems, such an invariant measure gives the probability distribution of chaotic orbits in the phase space, and it is intimately related to crucial mathematical concepts such as entropy and the Lyapunov exponent.

In 1960, a distinguished Polish-born mathematician Stan Ulam, father of the American hydrogen bomb, proposed a numerical method in his famous book entitled "A Collection of Mathematical Problems" to calculate a fixed density function of a Frobenius-Perron operator associated with a nonlinear mapping $S : [0,1] \to [0,1]$. He divided the interval $[0,1]$ into n subintervals $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. Let $I_i = [x_{i-1}, x_i]$ be the i-th subintervals, $i = 1, 2, \ldots, n$. Next, he defined an $n \times n$ nonnegative matrix $\bar{P}_n = [p_{ij}]$, in which the (i, j) element is $p_{ij} = m(I_i \cap S^{-1}(I_j))/m(I_i)$, where m is the Lebesgue measure. p_{ij} quantifies the fraction of those points in the i-th subinterval I_i that are mapped into the j-th subinterval I_j under S. In Ulam's method, one computes a normalized nonnegative left eigenvector v_n of $\bar{P}_n$ corresponding to eigenvalue 1, so that the corresponding piecewise constant function f_n with function values on each subinterval given by the components of v_n is a density function. This density function f_n can be considered as an approximate fixed density function of the Frobenius-Perron operator P. For the convergence of this numerical scheme based on a probability argument, Ulam presented his famous conjecture: if P has a fixed density function, then f_n approaches to a fixed density function f^* of P as n approaches infinity.

In 1973, Polish Academician Andrzej Lasota and Yorke solved another problem that Ulam proposed in "A Collection of Mathematical Problems", which is now a classic paper on the existence problem of the Frobenius-Perron operator. Ulam's problem is: if $S : [0,1] \to [0,1]$ is a sufficiently "simple" mapping (for example, a piecewise linear mapping or a polynomial mapping) such that the absolute value of its derivative is not less than 1, then does the corresponding Frobenius-Perron operator must have a fixed density function. In fact, Lasota and Yorke proved the following existence theorem: if $S : [0,1] \to [0,1]$ is a piecewise C^2 mapping such that the infimum of the absolute value of its derivative is greater than 1, then the existence of a fixed density function of the corresponding Frobenius-Perron operator is guaranteed, and every fixed density function is of bounded variation. The key to proving this theorem is using an inequality discovered by Yorke on a relation of variations between a function of bounded variation and its restriction to some subinterval. For the given mapping S, the Yorke inequality implies that a positive constant C exists such that for all functions f of bounded variation, there holds the following Yorke inequality

$$\bigvee_0^1 Pf \leq \frac{2}{\inf |S'(x)|} \bigvee_0^1 f + C \int_0^1 |f(x)| dx.$$

When Tien-Yien Li read the aforementioned Lasota-Yorke theorem, he keenly sensed that the concept of functions of bounded variation and the theorem by E. Helly on the sequence of functions of bounded variation must play a key role in proving the convergence of Ulam's method, and he firmly believed that Ulam's conjecture should stand for the above Lasota-Yorke class of interval mappings. He immediately began to study Ulam's numerical method. First, he defined a finite dimensional discretization operator Q_n associated with the partition $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ of the interval $[0, 1]$. The operator Q_n maps each integrable function f into a piecewise constant function that takes the average value of f on each $I_i = [x_{i-1}, x_i]$ as its value on I_i. Moreover, Q_n is not only a Galerkin projection operator that projects the L^1-space onto the subspace of piecewise constant functions, but also a Markov operator that preserves the integral of functions. If we compose Q_n with the Frobenius-Perron operator P to form $P_n = Q_nP$, then the matrix representation of P_n restricted to the subspace Δ_n consisting of all piecewise constant functions under the canonical density functions basis is exactly that very row stochastic matrix defined in Ulam's method. Utilizing Brouwer's fixed point theorem, Tien-Yien Li directly proved that for every natural number n, P_n has a piecewise constant fixed density function, and with the help of the Yorke inequality and Helly's theorem, he proved Ulam's conjecture for the Lasota-Yorke class of interval mappings. Specifically, he proved that the sequence of approximate fixed densities f_n produced from Ulam's method converges in L^1-norm to a fixed density function f^* of the Frobenius-Perron operator.

In the following twenty years, the computation of invariant measures has become an active branch of ergodic theory and nonlinear analysis. In almost all of the literature related to Ulam's method and its variants for the computation of invariant measures, this paper by Tien-Yien Li, published in the Journal of Approximation Theory, became one of the most essential and most widely cited papers. In addition, his thought process inspired his student Jiu Ding and Ding's collaborator Ai-Hui Zhou to prove the convergence of Ulam's method for the P. Góra-A. Boyarsky class of multi-dimensional piecewise expanding transformations in 1996.

3 Modern Homotopy Methods

Anyone who has studied algebraic topology or nonlinear functional analysis would know the famous Brouwer's fixed point theorem: a smooth mapping g from an n-dimensional closed ball D^n into itself must have a fixed point. A cunning way to prove this theorem is by contradiction. Suppose g has no fixed point. Then for each $x \in D^n$ let $f(x)$ be the intersection of the line segment from $g(x)$ to x extended to the sphere. It is easy to see that $f(x) = x$ if x is on the sphere. Thus, we obtain a smooth mapping from the closed ball D^n onto its boundary S^n such that its restriction to S^n is an identity mapping. However, differential topology tells us this is impossible.

In 1973, while Tien-Yien Li audited Professor Bruce Kellogg's graduate-level course "Numerical Solutions of Nonlinear Equations" at the University of Maryland and heard the above proof of Brouwer's fixed point theorem's published by Morris W. Hirsh in 1963, a marvelous idea emerged: In Hirsh's proof by contradiction, if g were to have no fixed point at all, then for the mapping f defined above, for almost all $y \in S^n$ the smooth curve $f^{-1}(y)$ would have no place to reach. Thus, g must have a fixed point. However, if we admit g has fixed points in the first place, f can still be defined except on those fixed points of g. Apparently for $y \in S^n$, $f^{-1}(y)$ must go toward the set of fixed points of g. More precisely, let F be the nonempty set of all fixed points of the smooth mapping $g : D^n \to D^n$, we can define a smooth mapping $f : D^n\backslash F \to S^n$ from the n-dimensional manifold $D^n\backslash F$ to the $(n-1)$-dimensional sphere S^n. From Sard's theorem of differential topology, y is a regular value of f for almost all $y \in S^n$. It follows that the inverse image of y under f, $f^{-1}(y)$, is a one dimensional manifold starting from y, that is, $f^{-1}(y)$ is a smooth curve. The other end of this curve can neither come back to the sphere nor stop inside $D^n\backslash F$. Therefore it must approach the fixed point set F of g. If this curve can be numerically followed, a fixed point of g can be calculated. Under the encouragement of Professors Kellogg and Yorke, Tien-Yien Li began to implement this idea on computer.

In the next three months, he spent nearly everyday with a computer for which the data could only be inputted with cards, each time without success. The stacks of paper that the computer spit out foreshadowed the program's failure. Tien-Yien Li was not defeated; he persevered in modifying the program. He modified and fixed, taking small steps from a computing novice down the path to expertise. At last, he beheld a single sheet of output from the computer, and on that sheet was a successful computation of a Brouwer's fixed point! He finally made it! Thus, a new numerical method for computing Brouwer's fixed points was born. It also paved the way for the modern homotopy continuation method.

The classical homotopy continuation method had been emerged as early as in the 1950s. In particular, D. Davidenko of the former Soviet Union introduced a corresponding initial value problem of an ordinary differential equation to numerically solve a homotopy equation. If we want to compute a zero point of a nonlinear mapping $f : R^n \to R^n$, we can construct a homotopy that combines f with a trivial mapping $f_0 : R^n \to R^n$ whose zero point x_0 is known (say, $f_0(x) = x - x_0$). More specifically, we define a homotopy mapping $H(x,t) = (1-t)f_0(x) + tf(x)$ with the parameter $0 \le t \le 1$. The traditional idea of the homotopy algorithm is based on the assumption that the zero points set $H^{-1}(0)$ of H can be represented as a curve $(x(t),t) \in R^n \times [0,1]$, $0 \le t \le 1$, that connects x_0 and a zero point x^* of f. Differentiating the identity $H(x(t),t) \equiv 0$ with respect to t, we obtain the initial value problem of the Davidenko ordinary differential equation: $x'(t) = -H_x(x,t)^{-1}H_t(x,t)$, $x(0) = x_0$, that can be numerically solved. By numerically integrating the above initial value problem from $t = 0$ to $t = 1$, a zero point x^* of f can be found. However, this method has a fatal weakness: in general, the homotopy curve $x(t)$ of $H^{-1}(0)$ may not always be monotonic in t. In other words, it may *turn around* with respect to t and at the turning point where $x'(t) = 0$, $H_x(x,t)^{-1}$ does not exist. The revolutionary idea of Kellogg-Li-Yorke applied to the homotopy method is: as long as 0 is a regular value of the homotopy mapping $H(x,t)$, by means of implicit function theorem, the smooth homotopy curve must exist, and in this case the coordinate vector variable x and the parameter variable t possess the same role. They may both be viewed as functions of the curve's arc length s for instance. Therefore, regardless of whether the curve "turns back" with respect to t or not, one can numerically follow the homotopy curve and find a solution by using the predictor-corrector technique. This is an important application of modern theoretical mathematics, especially differential topology, to the field of computational mathematics.

Interestingly, Kellogg-Li-Yorke's calculation of Brouwer's fixed point was not the first time it was done. They did not know that in 1967, Yale University's economics professor H. Scarf reduced the equilibrium point for a model in quantity economics to a fixed point problem of a continuous mapping f from an n-dimensional standard simplex into itself. According to the Brouwer's fixed point theorem, such a fixed point does exist. Scarf used the so-called simplicial triangulation of the simplex and then utilized Lemke's complementarity pivoting procedure and eventually leads to an approximate fixed point, resulting in a simplicial fixed point algorithm. In the seventies, this algorithm was extended to a class of simplicial algorithms to solve systems of nonlinear equations, which became a hot research topic during that period. In 1974, when the organizing committee of the First International Conference on Computing Fixed Point with Applications held at Clemson University found out Kellogg-Li-Yorke's new method, the committee immediately provided them with two airline tickets so that they may report their findings at the conference. As Scarf wrote in the Introduction of the conference proceedings "Fixed Point Algorithms and Applications," " For many of us one of the great surprises of the conference at Clemson was the paper by Kellogg, Li and Yorke which presented the first computational method for finding a fixed point of a continuous mapping making use of the considerations of differential topology instead of our customary combinatorial techniques. $\cdots$ " Today, Kellogg, Li, and Yorke together are widely regarded as the originators of the modern homotopy method for solving nonlinear problems, and they have contributed tremendously to this important field.

4 Solving Polynomial Systems

From the birth of the homotopy method for computing Brouwer's fixed point until today, Tien-Yien Li has been tirelessly excavating the way of solving polynomial systems. Solving the roots of polynomial systems is interesting and appears frequently in the scientific world, such as formula construction, geometric intersection, inverse kinematics, computation of equilibrium, etc. Meanwhile, these problems also arise in the research of chaos theory; for example, stationary solutions of the chaotic system of four ordinary differential equations studied by Lorenz are actually the solutions of the polynomial system in the right hand side.

Given a system of n polynomial equations with n variables, let d_i be the degree of the i-th polynomial for $i = 1, 2, \cdots, n$. Then the classic Bézout theorem in algebraic geometry gives an upper bound $d_1 d_2 \cdots d_n$ to the number of all isolated solutions of this system. This number is called the Bézout number associated with the polynomial system. Under most circumstances, this upper bound is much bigger than the actual number of isolated solutions. A typical example is the algebraic eigenvalue problem. The Bézout number of the quadratic polynomial system is 2^n corresponding to the eigenvalue problem of an $n \times n$ matrix A, but A has at most n eigenvalues.

In recent years, using the homotopy method to find *all* isolated solutions of a polynomial system has attracted much attention. In 1979, C. B. Garcia and W. I. Zangwill first established a homotopy $H : C^n \times [0, 1] \to C^n,\ H(x, t) = (1 - t)Q(x) + tP(x)$ for solving a polynomial system of n equations with n variables $P(x) = (p_1(x), p_2(x), \cdots, p_n(x)) = 0$. Here $Q = (q_1, q_2, \cdots, q_n)$ and each of its component functions $q_j : C^n \to C$ is defined as $q_j(x_1, x_2, \cdots, x_n) = x_j^{d_j+1} - 1$, where d_j is the degree of p_j. They proved that if 0 is a regular value of H, then each isolated solution of $P(x) = 0$ is an end point of a corresponding solution curve $x(t)$ to the homotopy equation $H(x, t) = 0$ at $t = 1$. The important fact is the curve $x(t)$ never turns around. Thus, we can solve the initial value problem of the ordinary differential equation $x'(t) = -H_x(x(t), t)^{-1}H_t(x(t), t),\ x(0) = x_0$, where x_0 is a solution of $Q(x) = 0$. In doing so, we can numerically follow $x(t)$ and find all the approximate isolated solutions. By Bézout's theorem, there are at most $d = d_1 d_2 \cdots d_n$ isolated solutions to $P(x) = 0$ while $Q(x) = 0$ has $d' = (d_1 + 1)(d_2 + 1) \cdots (d_n + 1)$ solutions. Thus, in order to find all isolated solutions of $P(x) = 0$, we must approximate d' different $x(t)$ curves. When $t \to 1$, many of those curves go to infinity and it is a big waste to trace all of those curves.

One advantage of the homotopy method for computing all the isolated solutions of polynomial systems is its parallelism, since one can solve the same ordinary differential equation with different initial values on a parallel machine. In order to overcome the inadequacies of Garcia-Zangwill's homotopy method mentioned above, S. N. Chow, J. Mallet-Paret, and Yorke introduced another homotopy $H(x, t) = (1 - t)Q(x) + tP(x) + t(1 - t)R(x)$, in which $q_j = x_j^{d_j} - b_j$ and $r_j = \sum_{i=1}^{n} a_{ij} x_i^{d_j},\ j = 1, \cdots, n$. They proved that, except for a set of measure 0, for all $(a, b) \in C^{n^2} \times C^n$, following all the d solution curves to the homotopy equation $H(x, t) = 0$ would find all the isolated solutions to $P(x) = 0$.

At the beginning of the 1980s, Tien-Yien Li greatly improved the construction of the homotopy mapping. He proved that for the initial polynomial system $q_j(x) = a_j x_j^{d_j} - b_j = 0,\ j = 1, 2, \cdots, n$, for almost all $(a, b) \in C^n \times C^n$, following all the d solution curves emanating from the solutions of $Q(x) = 0$ of the homotopy equation $H(x,t) = (1-t)Q(x) + tP(x) = 0$, one can find all the isolated solutions to $P(x) = 0$. In the following years, Tien-Yien Li continued to search for efficient numerical methods for solving polynomial systems for which the number of isolated solutions is much less than the Bézout number. This type of system is called the *deficient* system. If the usual homotopy method is used to solve this deficient polynomial system, we must follow d curves from $t = 0$ on, and when $t \to 1$, the majority of the curves go to infinity and only a small fraction of them would converge. This is a great waste of computation time.

For the most important and most often seen deficient polynomial system in numerical linear algebra - the matrix eigenvalue problem, Tien-Yien Li, his collaborators, and his students proposed the homotopy idea to numerically compute all the eigenvalues of a large scaled matrix A: Using a same order matrix D with all eigenvalues known or easy to get to, construct the homotopy $H(t) = (1-t)D + tA$, and then numerically follow the eigenvalue and eigenvector curves of $H(t)$ from $t = 0$ until the eigenvalues and eigenvectors of A are reached at $t = 1$. He and his Korean PhD student Noah Rhee were the first ones to apply this idea on the computer. Afterwards, he directed his Chinese students Hong Zhang, Kui-Yuan Li, Zhong-Gang Zeng, Liang-Jiao Huang, Luan Cong, and Min Jin to perfect this computational method. They have successfully developed various homotopy algorithms for computing eigenvalues and eigenvectors for real symmetric matrices, general real matrices, and large scaled sparse matrices. Even without taking consideration of the advantage of parallelism, the sequential homotopy algorithm outperforms some standard algorithms based on the QR decomposition for many large scaled algebraic eigenvalue problems.

For a general deficient polynomial system, the construction of a good homotopy algorithm largely depends on a smart choice of the initial polynomial system. This is because not only each isolated solution of the polynomial system $P(x) = 0$ is reached via a homotopy curve starting from a solution of the initial polynomial system $Q(x) = 0$, but more importantly we wish as few homotopy curves as possible would go to infinity as $t \to 1$. An ideal construction is such that $Q(x)$ and $P(x)$ have the same number of isolated zeros at infinity. In the last twenty years, Li, Sauer, Yorke, and his students Xiao-Shen Wang, Xin Li, and Tang-An Gao used the theory and methods from algebraic geometry to propose some powerful methods for choosing $Q(x)$, for example, the random product homotopy method and Cheater homotopy method. In the past decade, due to the application of D. N. Bernshtein's theorem, the polyhedral homotopy method based on the combinatorial counting of solutions has attracted great attention. In this new method, the computation of the so-called "mixed volume" is extremely important. Tien-Yien Li and his students, past and present, have obtained a series of outstanding new results in this area, and the details can be seen in his extensive survey paper published recently. In the field of solving polynomial systems, Tien-Yien Li well deserves a leading role.

5 Overcoming Obstacles

Incredulously, Tien-Yien Li's major contributions in the last thirty years were made while constantly combating excruciating pain. When he was an undergraduate at National Tsinghua University in Taiwan, his nickname was "baton." Apart from being one of the top academic students, he also excelled in athletics. He was once a member of the school's soccer team and captain of the basketball team. In the second year of working toward his PhD at the University of Maryland, however, he experienced gradual kidney failure. This did not hamper his unusual diligence, and he received his doctorate degree after completing eight papers in 1974. The sixth week after graduation, he discovered that his blood pressure was as high as 220/160. He began a five-and-a-half-year kidney dialysis on May 4, 1976, three times a week, five hours each time, not including time on the road. At the time, he conducted most of his research on his sickbed. On January 29, 1980, Tien-Yien Li underwent his first kidney transplant, but due to bodily rejection, the procedure was a failure. On July 15 of the following year, he successfully received a kidney from his sister. In the next three years, he recovered well. Good health did not linger, however; on February 21, 1984, Tien-Yien Li suffered from a stroke. As a result, the right half of his body was paralyzed, and he underwent a major brain surgery to settle a brain aneurysm on April 26. His health was stable afterwords. He did not have any more major surgeries in the next eight years, but minor operations still occurred often. Despite his ailments, Tien-Yien Li seized the opportunity to develop the important theory of homotopy continuation methods for solving eigenvalue problems and polynomial systems, and he trained a group of PhD students from mainland China. In addition to giving lectures in Taiwan, he also visited more than ten universities and research institutes of the Chinese Academy of Sciences in mainland China from June to July, 1985, giving numerous lectures on chaotic dynamical systems and homotopy methods. He began to accept graduate students from China, and has been devoted to training young Chinese scholars in mathematics since then.

On January 25, 1993, when Tien-Yien Li was teaching at Michigan State University, he experienced physical discomfort and was taken to the hospital in an unconscious state. The doctor diagnosed him with the blocking of brain artery. His will-power eventually defeated his disease. Beginning in 1992, he suffered from leg pains, and neither Western nor Eastern practitioner could explain the source of his distress. It was found out latter the pain was caused by spinal column arthritis, and a major surgery on May 30, 1995 severed the inflamed part. In the next five to six years, he enjoyed relatively good health. Yet in the first year of the current century, he underwent another spinal column surgery. Although his leg sporadically bothered him, he embarked on the journey to recovery in 2003. In the recent years, he devoted time to fitness and exercise, swimming one thousand yards or walking two miles each day. Consequently, his health has improved dramatically. When this biography was written in June of 2003, however, Tien-Yien Li suffered another attack. On June 24, the doctor successfully treated his clogged heart coronary artery with stents.

In the past few decades, Tien-Yien Li suffered from many illnesses, yet he fought against sickness and pain with all his might, time and again defeating them with his optimistic spirit. Up until now, he has gone through ten major operations and countless minor ones. His body is covered with surgical scars. He is a person who rises above the current situation, who does not give up without putting up a good fight, and who puts forth the best in the harshest environment. He often tells his graduate students that if they think of how he overcame excruciating pain when they encounter obstacles in studying and research, then they will also have the courage to overcome their own hindrances. It is this indefatigable spirit that enabled Tien-Yien Li to efficiently work under the continuous financial support from the National Science Foundation in the United States despite constantly being plagued by illnesses.

6 The Way to Success

Tien-Yien Li holds a strict attitude when it comes to academics. He believes that his success, apart from having an excellent advisor like Professor Yorke, stems mainly from perseverance. He often tells his students that he himself is not smart, and that being smart is not as important as being able to dig down to the root of the problem. He emphasizes that he merely spends one more minute on a problem than do his peers. That precious minute may well be the minute leading to success. A big shot not being able to solve a problem does not mean that a small fish also cannot, and a big shot's train of thought does not mean it will lead to a solution. "Endure; persevere; do not give up" is the maxim that he shares with his students. He also says that one who studies must understand the material thoroughly, especially when the subject is mathematics; vaguely memorizing logical procedures is useless. He once gave this example: why the row rank of a matrix equals its column rank? Anyone who has taken Linear Algebra can prove it. But what is its geometric significance? What is its significance in physics? If you look at the problem from many different angles, then you will reap surprising rewards.

Tien-Yien Li attended college in Taiwan, thus he is familiar with, and strongly opposed to, the general method of pure memorization in Chinese institutions of higher learning. He once told the following story: when a graduate student took the oral part of the mathematics qualifying exam, the professor wanted to test her on a special case of the Tychonoff theorem in Topology: the product of two compact sets is compact. She begged the professor to let her prove the general Tychonoff theorem: the product of any number of compact sets is compact, because she remembered every detail of that general proof, yet she did not know how to prove the simpler case with only two compact sets. Tien-Yien Li is strongly opposed to pure memorization without true understanding. All the graduate students who have participated in his mathematics seminars will not forget his basic requirement: Do not just speak with the "$\epsilon - \delta$" language; that is merely logic. If you speak, speak about the "basic idea." He requires his students to clearly explain concrete or unusual cases before demonstrating a proof to a general theorem and not play hide-and-go-seek with abstract concepts. He firmly believes that if a person truly understands a course, then he can explain it in a way that even an average person can understand. He also practices what he believes. He always started his invited mathematics lectures throughout the world from the most elementary concept, and his audiences were attracted by his vivid and thoughtful talks. He also uses this standard to train his students. In 1986, when he let his new Chinese student report the paper "Rudiments of an average case complexity theory for piecewise-linear path following algorithms" by S. Smale's student J. Renegar (now a full professor at Cornell University), his first sentence was, "You must pretend that I am ignorant of anything." At that time, the student was puzzled: the famous professor, because of whom he applied to Michigan State University, claims that he is "ignorant." It is facing this "ignorant" mathematician, this student learned what is researching mathematics, what is talking about mathematics.

Because of Tien-Yien Li's unique research and teaching methods, he not only received Michigan State University's Distinguished Professorship and Distinguished Faculty awards, but he also encouraged his graduate students to pursue research and teaching simultaneously. His course of erudition is truly an inspiration to the growth of a mathematician.

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About the Article

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