Since $f(n)$ and $g(n)$ are asymptotically non-negative functions, there exists $n_0$ such that $f(n) \geq 0 \quad \forall \ n \geq n_0$.

For such $n_0$, $\max(f(n), g(n)) \leq f(n) + g(n) \quad \forall \ n \geq n_0$.

Also, for $n_0$, $\frac{1}{2} [f(n) + g(n)] \leq \max(f(n), g(n)) \quad \forall \ n \geq n_0$

since the max of two non-negative numbers is no less than their average.

Therefore, for the choices of $c_1 = \frac{1}{2}$, $c_2 = 1$ and $n_0 = n_0$ we have

$$c_1 [f(n) + g(n)] \leq \max(f(n), g(n)) \leq c_2 [f(n) + g(n)] \quad \forall \ n \geq n_0$$

$$\Rightarrow \quad \max(f(n), g(n)) = \Theta(f(n) + g(n)).$$
(2) Case \( a \geq 0 \): Here \( n + a = n - |a| \).

Now, \( n \leq n + |a| \leq 2n \), if \( n \geq |a| \)

\[ \Rightarrow n^b \leq (n + |a|)^b \leq 2^b n^b, \therefore b > 0. \]

\therefore \text{For the choices of } c_1 = 1, c_2 = 2^b \text{ and } n_0 = |a|, \text{ we have}

\[ c_1 n^b \leq (n + a)^b \leq c_2 n^b \quad \forall n \geq n_0 \text{ in this case.} \]

\[ \Rightarrow (n + a)^b = \Theta(n^b) \]

Case \( a < 0 \): Here \( n + a = n - |a| \).

Now, \( \frac{1}{2}n \leq n - |a| \leq n \), if \( \frac{1}{2}n \geq |a| \)

\[ \text{i.e., } n \geq 2|a| \]

\[ \Rightarrow \left(\frac{1}{2}\right)^b n^b \leq (n - |a|)^b \leq n^b, \therefore b > 0. \]

\therefore \text{For the choices of } c_1 = (\frac{1}{2})^b, c_2 = 1 \text{ and }

\[ n_0 = 2|a|, \text{ we have} \]

\[ c_1 n^b \leq (n + a)^b \leq c_2 n^b \quad \forall n \geq n_0 \text{ in this case.} \]

\[ \Rightarrow (n + a)^b = \Theta(n^b). \]

Finally combining the cases, for the choices of

\[ c_1 = \min \left(1, \left(\frac{1}{2}\right)^b\right) = \left(\frac{1}{2}\right)^b, \]

\[ c_2 = \max (2^b, 1) = 2^b, \]

and \( n_0 = \max(|a|, 2|a|) = 2|a| \),

we have \( c_1 n^b \leq (n + a)^b \leq c_2 n^b \quad \forall n \geq n_0 \)

\[ \therefore (n + a)^b = \Theta(n^b). \]
(a) From Stirling's formula
\[ \lg(n!) = \frac{1}{2} \lg(2\pi) + \frac{1}{2} \lg n + n \lg \left( \frac{n}{e} \right) + \lg(1 + \Theta(\frac{1}{n})) \]
\[ = 1.326 + \frac{1}{2} \lg n + n \lg n - n \lg e + \lg(1 + \Theta(\frac{1}{n})) \]
\[ = 1.326 + \frac{1}{2} \lg n + n \lg n - 1.44 \cdot n + \lg(1 + \Theta(\frac{1}{n})). \]

We see that \( \lg(n!) \leq 2n \lg n \)
if \( n \lg n \geq 1.326 + \frac{1}{2} \lg n - 1.44 \cdot n + \lg(1 + \Theta(\frac{1}{n})). \)

This is true \( \forall n \geq 4. \)

In the same way, \( \lg(n!) \geq \frac{1}{2} n \lg n \)
if \( n \lg n \geq 2.88n - 2.652 - \lg n - 2 \lg(1 + \Theta(\frac{1}{n})) \)
which is also true for \( n \geq 4. \)

\[ \therefore \text{ For the choices of } c_1 = \frac{1}{2}, \ c_2 = 2, \ n_0 = 4, \ c_1 \ n \lg n \leq \lg(n!) \leq c_2 n \lg n \ \forall n \geq n_0. \]

\[ \therefore \lg(n!) = \Theta(n \lg n). \]

(b) \[ \lim_{n \to \infty} \left( \frac{n!}{2^n} \right) = \lim_{n \to \infty} \left[ \frac{\sqrt{2\pi n} \left( \frac{n}{e} \right)^n (1 + \Theta(\frac{1}{n}))}{2^n} \right] \]
\[ = \lim_{n \to \infty} \left[ \sqrt{2\pi n} \left( \frac{n}{2e} \right)^n (1 + \Theta(\frac{1}{n})) \right] \]
which goes to \( \infty \) since \( 2e \) is a constant.

\[ \therefore \ n! = \omega(2^n). \]

(c) \[ \lim_{n \to \infty} \left( \frac{n!}{n^n} \right) = \lim_{n \to \infty} \left[ \frac{\sqrt{2\pi n} \left( \frac{n}{e} \right)^n (1 + \Theta(\frac{1}{n}))}{n^n} \right] \]
\[ = \lim_{n \to \infty} \left[ \frac{\sqrt{2\pi n} \cdot \Theta(1 + \Theta(\frac{1}{n}))}{e^n} \right] \]
\[ = 0, \text{ since the numerator grows much slower than the denominator.} \]

\[ \therefore \ n! = o(n^n). \]

The limits in (b) and (c) can be established more rigorously using L'Hospital's rule.
(a) Since \( f(n) = \Theta(g(n)) \), \( f(n) \leq c \cdot g(n) \), \( \forall n \geq n_0 \)
for some \( c, n_0 > 0 \).

\[
\therefore \ lg(f(n)) \leq lg(c) + lg(g(n)), \text{ if } n_0 \text{ is large enough}
\]
so that \( f(n) \geq 1 \) i.e., \( lg(f(n)) \geq 0, \forall n \geq n_0 \).

Now, we are given that for large enough \( n \),
\( lg(g(n)) \geq 1 \). Choose \( n_0 \) large enough so that
\( lg(g(n)) \geq 1 \), \( \forall n \geq n_0 \).

For such an \( n_0 \),

\[
lg(f(n)) \leq lg(c) + lg(g(n))
\leq (lg(c) \cdot lg(g(n)) + lg(g(n)) \quad [\because lg(g(n)) \geq 1]
\]
\[
= lg(g(n)) (1 + lg(c))
\]

\[
\therefore \text{For the choice of } c' = (1 + lg(c)) \text{ and above } n_0, \quad lg(f(n)) \leq c' \cdot lg(g(n)) \quad \forall n \geq n_0.
\]

\[
\therefore \ lg(f(n)) = \Theta(lg(g(n))).
\]

This conjecture is true.

(b) This conjecture is false. The following is a counter example.
Choose \( f(n) = 2n \) and \( g(n) = n \).
Then \( f(n) \leq 2 \cdot g(n) \), \( \forall n \geq 1 \), i.e. \( f(n) = \Theta(g(n)) \).

But we know from exercise 3.1-4 that
\( 2^{2n} \neq \Theta(2^n) \); this was proved in class.

\[
\therefore \text{In this case } 2^{f(n)} \neq \Theta(2^{g(n)}).
\]
(c) Given $f(n) = O(g(n))$, that is $f(n) \leq c \cdot g(n) \quad \forall n \geq n_0$, for some $c, n_0 > 0$.

\[ g(n) \geq \left(\frac{1}{c}\right) f(n) \quad \forall n \geq n_0. \]

So, for the choices of $c' = \left(\frac{1}{c}\right)$ which is $> 0$, and the above $n_0$,

\[ g(n) \geq c' f(n) \quad \forall n \geq n_0, \]

i.e., $g(n) = \Omega(f(n))$.

This conjecture is true.

(d) Let $g(n) = o(f(n))$. Then, is $f(n) + g(n) = \Theta(f(n))$?

Since $g(n) = o(f(n))$,

\[ \lim_{n \to \infty} \left(\frac{g(n)}{f(n)}\right) = 0. \]

That is for any $c > 0$, there exists $n_{oc}$ such that $0 \leq g(n) < c \cdot f(n) \quad \forall n \geq n_{oc}$. Select any such $(c, n_{oc})$ pair.

Then

\[ f(n) + g(n) < f(n) + c \cdot f(n) \]

\[ = (1+c) \cdot f(n). \]

Also, $f(n) + g(n) \geq f(n)$, since $g(n) \geq 0$.

\[ \therefore \text{For the choices of } c_1 = 1, \ c_2 = (1+c) \text{ and } n_0 = n_{oc}, \text{ we have} \]

\[ c_1 \cdot f(n) \leq f(n) + g(n) \leq c_2 \cdot f(n) \quad \forall n \geq n_0. \]

\[ \therefore f(n) + o(f(n)) = \Theta(f(n)). \]

This conjecture is true.